# B. Math. Ist Year IInd Semester Final Examination

## Analysis II

**Total : 100** **04.05.2001** **Time: 3 hours**

Answer all questions. The maximum you can score is 100.

1. Let $(X, d)$ be a compact metric space. Given $\epsilon > 0$, prove that $\exists$ finitely many points, $x_1, x_2, \ldots, x_k$ with the property that : given $x \in X, \exists\ j, 1 \leq j \leq k$, with $d(x, x_j) < \epsilon$. (10)

2. $(X, d)$ is a metric space. $\{x_n\}$ is a Cauchy sequence in $X$ with a convergent subsequence $\{x_{n_k}\}$ and $x_{n_k} \to x \in X$. Prove that $\{x_n\}$ is itself convergent with $x_n \to x$. (10)

3. Consider $I\!R^n$ with the following metric : $d(x, y) = \frac{\|x-y\|}{1+\|x-y\|}$. Note that $d(x, y) < 1, \forall\ x, y \in I\!R^n$. Is $I\!R^n$ compact in this metric? Justify your answer. (10)

4. Let $f(x) = e^{e^{x^2+y^3}}$. Calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$. Quote precisely a theorem proved in class to explain why it is enough to calculate just one of them. (10)

5. (a) Suppose $f$ is a smooth function of $n$ variables $(x_1, x_2, \ldots, x_n) = x$. Let $x_0 = (x_{01}, \ldots, x_{0n})$. Write down the expression for the $k^{\text{th}}$-Taylor expansion in multi-index notation and also the expression for the remainder term. (10)

   (b) If $f(x_1, x_2, x_3) = x_1^3 x_2^3 x_3^3$, what is $\frac{\partial^\alpha f}{\partial x^\alpha}$, for $\alpha = (1, 2, 1)$. What is $\frac{\partial^\beta f}{\partial x^\beta}$, where $\beta = (0, 4, 0)$? (10)

6. What are the level sets of the function $f(x, y) = e^{x^2-y}$. Determine the direction of the greatest rate of increase of the function at (0,0). (10)

7. Find the maximum value of the function $f(x, y, z) = xy(1 - (x + y))$ in the region $x > 0, y > 0, \frac{1}{4} < x + y < 4$. At what point does this occur? Justify your answer. (10)

8. Let $\vec{f}(x, y, z) = \left(\frac{x-4}{[(x-4)^2+y^2+z^2]^{3/2}}, \frac{y}{[(x-4)^2+y^2+z^2]^{3/2}}, \frac{z}{[(x-4)^2+y^2+z^2]^{3/2}}\right)$. Prove that if $S$ is a closed surface not enclosing the points (4,0,0), then $\int_S \vec{F} \cdot d\vec{A} = 0$. What happens if $S$ encloses the point (4,0,0)? (15)

9. (a) Let $f(z) = \text{Re } z$. (Real part of $z$.) Prove that this is *not* a holomorphic function on any open set. (10)

(b) Calculate $\oint_r f(z)dz$ over the simple closed curve $r = \{z : |z| = 1\}$, where $f(z)$ is as in Part a). (5)